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Schlesinger transformations for the second members of PII and PIV hierarchies

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Abstract
In this paper, we give a method to obtain the Schlesinger transformations for the second members of second and fourth Painlevé hierarchies. The procedure involves formulating a Riemann–Hilbert problem for a transformation matrix which transforms the solution of the linear problem but leaves the associated monodromy data the same.

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1. Introduction

The six Painlevé equations, PI–PVI, are found by P Painlevé and B Gambier as the only irreducible second-order ordinary differential equations (ODEs) whose general solutions are free from movable critical points [1]. One of the important properties of the Painlevé equations is the existence of Schlesinger transformations [2–4], that is transformations that transform the solutions of the associated linear problem but preserve the monodromy data.

Recently there has been much interest in higher order analogues of the Pianlevé equations [5–13]. In [11], non-isospectral scattering problems have been used to derive new hierarchies of ordinary differential equations. These hierarchies are called second and fourth Painlevé hierarchies since they have the second and fourth Pianlevé equations, PII and PVI, as first members, respectively. Other second Painlevé hierarchies can be found in [12, 13].

In this paper, we present a method to obtain the Schlesinger transformations for the second members of the second and fourth Painlevé hierarchies given in [11]. These transformations lead to new Bäcklund transformations for these equations. Bäcklund transformations for the fourth Painlevé hierarchy given in [11] was also studied in [14].

2. The second member of the PII hierarchy

In [11], a PII hierarchy is given as follows:

$$\begin{pmatrix} \partial_x^{-1} & 0 \\ 0 & \partial_x^{-1} \end{pmatrix} \left[\mathcal{R}^n \mathbf{u}_x + \sum_{j=0}^{n-2} c_j \mathcal{R}^j \mathbf{u}_x \right] + \begin{pmatrix} g_{n+1} x \\ -\delta_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1}$$

where

$$\begin{aligned} \mathbf{u} &= (u, v)^T, \\ \mathcal{R} &= \frac{1}{2} \begin{pmatrix} \partial_x u \partial_x^{-1} - \partial_x & 2 \\ 2v + v_x \partial_x^{-1} & u + \partial_x \end{pmatrix}. \end{aligned} \tag{2}$$

The first member of the PII hierarchy (1), that is $n = 1$, is the PII equation.

The second member of PII hierarchy (1) reads

$$u_{xx} = 3uu_x - u^3 - 6uv - 4c_0 u - 4g_3 x, \tag{3a}$$

$$v_{xx} = -3uv_x - 3u^2 v - 3v^2 - 4c_0 v + 4\delta_2. \tag{3b}$$

Since $g_3 \neq 0$, without loss of generality we assume that $g_3 = 1$; and we set $\delta = \delta_2$. Eliminating v between (3a) and (3b), we get the following fourth-order equation for u

$$\begin{aligned} u_{xxxx} = {} & \frac{2u_x u_{xxx}}{u} + \frac{3(u_{xx})^2}{2u} - \left[\frac{2(u_x)^2}{u^2} - 5u^2 - \frac{8x}{u} \right] u_{xx} + \left[\frac{5u}{2} - \frac{8x}{u^2} \right] (u_x)^2 \\ & + \frac{8u_x}{u} - \frac{5}{2} u^5 - 12c_0 u^3 - 8xu^2 - 4\left(2c_0^2 + 6\delta + 3\right) u + \frac{8x^2}{u}. \end{aligned} \tag{4}$$

Equation (4) can be obtained as the compatibility condition of the following linear system of equations [15]:

$$\frac{\partial \Phi}{\partial \lambda} = B(\lambda) \Phi(\lambda), \tag{5a}$$

$$\frac{\partial \Phi}{\partial x} = A(\lambda) \Phi(\lambda), \tag{5b}$$

where

$$\begin{aligned} A &= \begin{pmatrix} -\lambda & \frac{w}{2} \\ -2v/w & \lambda \end{pmatrix}, \\ B &= B_3 \lambda^3 + B_2 \lambda^2 + B_1 \lambda + B_0, \\ B_3 &= -2\sigma_3, \qquad B_2 = \begin{pmatrix} 0 & w \\ -4v/w & 0 \end{pmatrix}, \\ B_1 &= \begin{pmatrix} -(v + 2c_0) & wu/2 \\ -2w^{-1}(v_x + uv) & (v + 2c_0) \end{pmatrix}, \\ B_0 &= \begin{pmatrix} -\left(\frac{1}{2} v_x + uv + x\right) & \frac{w}{4}(u^2 - u_x + 2v + 4c_0) \\ w^{-1}(v^2 + uv_x - vu_x + 2u^2 v - 4\delta) & \left(\frac{1}{2} v_x + uv + x\right) \end{pmatrix}, \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad u = -\frac{w_x}{w}. \end{aligned} \tag{6}$$

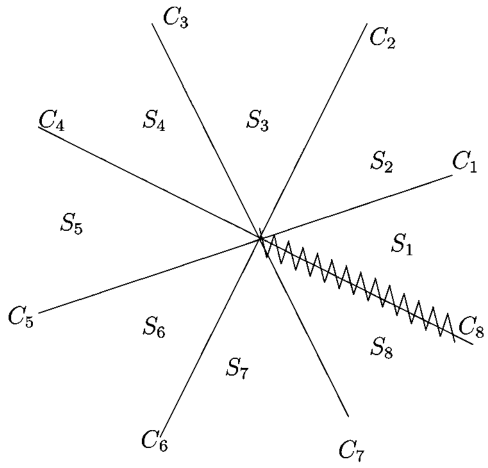

Figure 1.

2.1. Direct problem

The essence of the direct problem is to establish the analytic structure of Φ with respect to λ in the entire complex λ-plane. Since (5*a*) is a linear ODE in λ, the analytic structure is completely determined by its singular points. Equation (5*a*) has an irregular singularity at $\lambda = \infty$.

Solution about $\lambda = \infty$. The solution $\Phi(\lambda)$ of (5) in the neighbourhood of the irregular singular point $\lambda = \infty$ has the formal expansion

$$\Phi_\infty = \hat{\Phi}_\infty \lambda^{D_\infty} \mathrm{e}^{Q(\lambda)} = (I + \Phi_{\infty 1}\lambda^{-1} + \cdots)\lambda^{D_\infty} \mathrm{e}^{Q(\lambda)}, \tag{7}$$

where

$$D_\infty = \delta\sigma_3, \qquad Q(\lambda) = -\left(\tfrac{1}{2}\lambda^4 + c_0\lambda^2 + x\lambda\right)\sigma_3. \tag{8}$$

The actual asymptotic behaviour of Φ changes in certain sectors of the complex λ-plane. These sectors are determined by $\mathrm{Re}\left(\frac{1}{4}\lambda^4 + c_0\lambda^2 + x\lambda\right) = 0$; thus for large λ the sectors are asymptotic to the rays $\arg\lambda = \frac{\pi}{8}(2j-3)$, $j = 1, 2, \ldots, 8$. Let $\Phi_j(\lambda)$, $j = 1, 2, \ldots, 8$ be solutions of (36) such that $\det\Phi_j(\lambda) = 1$ and $\Phi_j(\lambda) \sim \Phi_\infty$ as $|\lambda| \to \infty$ in the sector $S_j : \frac{\pi}{8}(2j-3) \leqslant \arg\lambda < \frac{\pi}{8}(2j-1)$ (see figure 1). Then the solutions $\Phi_j(\lambda)$ are related by the Stokes matrices, G_j, as follows

$$\begin{aligned} &\Phi_{j+1}(\lambda) = \Phi_j(\lambda)G_j, \qquad j = 1, 2, \ldots, 7, \\ &\Phi_1(\lambda) = \Phi_8(\lambda\,\mathrm{e}^{2\pi\mathrm{i}})G_8\,\mathrm{e}^{-2\pi\mathrm{i}D_\infty}, \end{aligned} \tag{9}$$

where

$$G_{2j-1} = \begin{pmatrix} 1 & a_{2j-1} \\ 0 & 1 \end{pmatrix}, \qquad G_{2j} = \begin{pmatrix} 1 & 0 \\ a_{2j} & 1 \end{pmatrix}, \qquad j = 1, 2, 3, 4. \tag{10}$$

The monodromy data $\{a_j : j = 1, 2, \ldots, 8\}$ satisfies the consistency condition

$$\prod_{j=1}^{8} G_j = \mathrm{e}^{2\pi i D_\infty}. \tag{11}$$

2.2. Schlesinger transformations

Let $\Phi(\lambda)$ be solution of (5) with parameter δ and let $\Phi'(\lambda)$ be solution of (5) with parameters δ'. We consider transformation

$$\Phi'(\lambda) = R(\lambda)\Phi(\lambda) \tag{12}$$

such that $\Phi'(\lambda)$ and $\Phi(\lambda)$ have the same monodromy data. Let $\delta' = \delta + n$. Then (11) is invariant if $n \in \mathbb{Z}$.

Let $R(\lambda) = R_j(\lambda)$ when $\lambda \in S_j$, $j = 1, 2, \ldots, 8$. Then (9) implies the following RH problem for $R(\lambda)$

$$\begin{aligned} R_{j+1}(\lambda) &= R_j(\lambda), \qquad \lambda \text{ on } C_{j+1}, \qquad j = 1, 2, \ldots, 7, \\ R_1(\lambda) &= R_8(\lambda\, e^{2\pi i}), \qquad \lambda \text{ on } C_1, \end{aligned} \tag{13}$$

with the boundary conditions

$$R(\lambda) \sim \hat{\Phi}'_\infty \lambda^{n\sigma_3} \hat{\Phi}_\infty^{-1}, \qquad \text{as } |\lambda| \to \infty. \tag{14}$$

All possible Schlesinger transformations admitted by equation (3) may be generated by the following transformations:

$$\delta' = \delta + 1, \qquad R_{(1)}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} -\frac{u}{2} & -\frac{w}{4} \\ \frac{4}{w} & 0 \end{pmatrix}, \tag{15}$$

$$\delta' = \delta - 1, \qquad R_{(2)}(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \lambda + \begin{pmatrix} 0 & \frac{w}{v} \\ -\frac{v}{w} & -\frac{1}{2v}(v_x + uv) \end{pmatrix}. \tag{16}$$

Let

$$\Phi'(\lambda, x; u', v', \delta') = R_{(1)}(\lambda, x; u, v, \alpha, \beta)\Phi(\lambda, x; u, v, \delta), \tag{17}$$

and

$$\Phi''(\lambda, x; u'', v'', \alpha'', \beta'') = R_{(2)}(\lambda, x; u', v', \alpha', \beta')\Phi'(\lambda, x; u', v', \alpha', \beta'). \tag{18}$$

Then

$$R_{(2)}(\lambda, x; u', v', \delta')R_{(1)}(\lambda, x; u, v, \delta) = I. \tag{19}$$

2.3. Bäcklund Transformations

The linear equation (5a) is transformed under the Schlesinger transformations defined by the transformation matrices $R_{(j)}$, $j = 1, 2$ as follows:

$$\frac{\partial \Phi'}{\partial \lambda} = B'(\lambda)\Phi'(\lambda), \tag{20a}$$

$$B'(\lambda) = \left[R_{(j)}(\lambda)B(\lambda) + \frac{\partial}{\partial \lambda} R_{(j)}(\lambda) \right] R_{(j)}^{-1}(\lambda). \tag{20b}$$

Using (20b) we can derive the Bäcklund transformations between solutions $u(x)$ and $v(x)$ of (3), with parameters δ and solutions $u'(x)$ and $v'(x)$ of (3), with parameters δ'. The Bäcklund transformations corresponding to the Schlesinger transformations $R_{(j)}$, $j = 1, 2$ may be listed as follows:

$$\begin{aligned} R_{(1)} : \quad & v' = v - u_x, \\ & u' = \frac{1}{(v - u_x)}[2uu_x - v_x - u^3 - 5uv - 4c_0 u - 4x], \\ & \delta' = \delta + 1, \end{aligned} \tag{21}$$

$$
\begin{aligned}
R_{(2)}: \quad & v' = -\left[\left(\frac{v_x}{v}+u\right)^2 + u\left(\frac{v_x}{v}+2u\right) + 2v + 4c_0 - u_x - \frac{4\delta}{v}\right],\\
& u' = \left(\frac{v_x}{v}+u\right),\\
& \delta' = \delta - 1.
\end{aligned} \tag{22}
$$

2.4. Special solutions

It is well known that for certain choice of parameters, PII–PVI admit special solutions that are either rational functions or can be expressed by the classical transcendental functions [16]. In this section, we will study special solutions of (4).

The Bäcklund transformation (21) breaks down if $v - u_x = 0$ and $2uu_x - v_x - u^3 - 5uv - 4c_0u - 4x = 0$. Eliminating v between these two equations, we obtain

$$
u_{xx} + 3uu_x + u^3 + 4c_0u + 4x = 0. \tag{23}
$$

However u and v satisfy (3). This implies that δ must satisfy $\delta + 1 = 0$. Equation (23) is equation PVI in the complete list of second-order Painlevé equations (see [1] page 334). Therefore we have shown that if $\delta = -1$, then (4) admits special solution $u = \frac{y_x}{y}$, where y is a solution of the linear equation

$$
y_{xxx} = -4c_0y_x - 4xy. \tag{24}
$$

Another special solution of (4) can be obtained from the Bäcklund transformation (22). This transformation breaks down when $v = \delta = 0$. Substituting these values in (3), we obtain

$$
u_{xx} - 3uu_x + u^3 + 4c_0u + 4x = 0. \tag{25}
$$

The solution of (25) is given by $u = -\frac{y_x}{y}$, where y is a solution of the linear equation

$$
y_{xxx} = -4c_0y_x + 4xy. \tag{26}
$$

One can use the transformation (21) and (22) to obtain infinite hierarchies of solutions of (4). For example, if we apply the transformation (22) to the solution

$$
u = \frac{y_x}{y}, \qquad \delta = -1, \tag{27}
$$

where y satisfies (24), then we obtain the new solution

$$
u' = -\frac{2y_x}{y} - \frac{1}{y\left(yy_{xx} - y_x^2\right)}\left[y_x^3 + 4c_0y^2y_x + 4xy^3\right], \qquad \delta' = -2. \tag{28}
$$

Applying the transformation (21) to the solution

$$
u = -\frac{y_x}{y}, \qquad \delta = 0, \tag{29}
$$

where y satisfies (26), we obtain the new solution

$$
u' = -\frac{2y_x}{y} + \frac{1}{y\left(yy_{xx} - y_x^2\right)}\left[y_x^3 + 4c_0y^2y_x - 4xy^3\right], \qquad \delta' = 1. \tag{30}
$$

3. The second member of the PIV hierarchy

In this section, we consider the second member of a PIV hierarchy given in [11] as follows:

$$\begin{aligned} L_{n,x} &= 2K_n + uL_n + g_n - 2\alpha_n, \\ K_{n,x} &= \frac{1}{L_n}\left[\left(K_n + \frac{1}{2}g_n - \alpha_n\right)^2 - \frac{1}{4}\beta_n^2\right] - vL_n, \end{aligned} \tag{31}$$

where $\mathbf{K}_n = (K_n, L_n)^T$ is defined recursively as follows:

$$\begin{aligned} &\mathbf{K}_n[\mathbf{u}] = \mathbf{L}_n[\mathbf{u}] + \sum_{j=1}^{n-1} c_j \mathbf{L}_j[\mathbf{u}] + g_n x \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ &\mathbf{u} = (u, v)^T, \qquad \mathbf{L}_1[\mathbf{u}] = (v, u)^T, \\ &\mathcal{B}_1 \mathbf{L}_{j+1}[\mathbf{u}] = \mathcal{B}_2 \mathbf{L}_j[\mathbf{u}], \\ &\mathcal{B}_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \qquad \mathcal{B}_2 = \frac{1}{2}\begin{pmatrix} 2\partial_x & \partial_x u - \partial_x^2 \\ u\partial_x + \partial_x^2 & v\partial_x + \partial_x v \end{pmatrix}. \end{aligned} \tag{32}$$

The first member of PIV hierarchy (31), that is $n = 1$, is the PIV equation.

The second member of PIV hierarchy (31) reads

$$u_{xx} = 3uu_x - u^3 - 6uv - 2g_2 xu + 2c_1(u_x - 2v - u^2) + 4\alpha_2, \tag{33a}$$

$$\begin{aligned} v_{xx} = &\frac{(2uv + v_x + 2c_1 v - 2\alpha_2 + g_2)^2 - \beta_2^2}{(2v + u^2 - u_x + 2g_2 x + 2c_1 u)} \\ &- 2(uv)_x - 2c_1 v_x - v[2uv + v_x + 2c_1 v - 2\alpha_2 + g_2]. \end{aligned} \tag{33b}$$

Without loss of generality we assume that $g_2 = 1$; and we set $\alpha = \alpha_2$ and $\beta = \beta_2$. Eliminating v between (33*a*) and (33*b*), we get the following fourth-order equation for u:

$$T_{xx} = \frac{1}{2T}(T^2 - \beta^2) - 2T^2 + T\left[\frac{3}{2}u^2 + 2c_1 u + 2x\right], \tag{34}$$

where

$$T = \frac{-1}{2(3u + 2c_1)}[u_{xx} - 2(u + c_1)(u^2 + 2c_1 u + 2x) - 4\alpha]. \tag{35}$$

Equation (34) can be obtained as the compatibility condition of the following linear system of equations [15]

$$\frac{\partial \Phi}{\partial \lambda} = B(\lambda)\Phi(\lambda), \tag{36a}$$

$$\frac{\partial \Phi}{\partial x} = A(\lambda)\Phi(\lambda), \tag{36b}$$

where

$$
\begin{aligned}
&A = \begin{pmatrix} -\lambda & w \\ -v/w & \lambda \end{pmatrix}, \\
&B = B_2\lambda^2 + B_1\lambda + B_0 + B_{-1}\lambda^{-1}, \\
&B_2 = -2\sigma_3, \qquad B_1 = \begin{pmatrix} -2c_1 & 2w \\ -2v/w & 2c_1 \end{pmatrix}, \\
&B_0 = \begin{pmatrix} -(v+x) & w(u+2c_1) \\ -w^{-1}(v_x + uv + 2c_1 v) & (v+x) \end{pmatrix}, \\
&B_{-1} = \begin{pmatrix} -H & wL \\ -\frac{\left(H^2 - \frac{1}{4}\beta^2\right)}{wL} & H \end{pmatrix}, \\
&\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
&u = -\frac{w_x}{w}, \qquad L = \frac{1}{2}[2v + u^2 - u_x + 2x + 2c_1 u], \\
&H = \frac{1}{2}[v_x + 2uv + 2c_1 v - 2\alpha + 1].
\end{aligned}
\tag{37}
$$

3.1. Direct Problem

The equation (36*a*) has a regular singularity at $\lambda = 0$ and an irregular singularity at $\lambda = \infty$.

Solution about $\lambda = 0$. It is well known that the solution of a linear ODE in the neighbourhood of an isolated regular singular point can be obtained via a convergent power series. In this particular case, the solution $\Phi(\lambda)$ of (36) in the neighbourhood of the regular singularity at $\lambda = 0$, for $\beta \notin \mathbb{Z}$, has the form

$$\Phi_0(\lambda) = \hat{\Phi}_0\lambda^{D_0} = G_0(I + \Phi_{01}\lambda + \Phi_{02}\lambda^2 + \cdots)\lambda^{D_0}, \tag{38}$$

where

$$
\begin{aligned}
&D_0 = \frac{1}{2}\beta\sigma_3, \\
&G_0 = \begin{pmatrix} wL\rho_1 & wL\rho_2 \\ \left(H + \frac{1}{2}\beta\right)\rho_1 & \left(H - \frac{1}{2}\beta\right)\rho_2 \end{pmatrix}, \qquad \det G_0 = 1, \\
&\rho_1 = \kappa_1 \exp\left[-\int^x \frac{\left(H - \frac{1}{2}\beta\right)}{L}\,\mathrm{d}x'\right], \qquad \kappa_1 = \text{constant}, \\
&\rho_2 = \kappa_2 \exp\left[-\int^x \frac{\left(H + \frac{1}{2}\beta\right)}{L}\,\mathrm{d}x'\right], \qquad \kappa_2 = \text{constant},
\end{aligned}
\tag{39}
$$

and Φ_{01} satisfies the equation

$$\Phi_{01} + [\Phi_{01}, D_0] = G_0^{-1} B_0 G_0. \tag{40}$$

The monodromy matrix about $\lambda = 0$ is given as

$$\Phi_0(\lambda\,\mathrm{e}^{2\pi i}) = \Phi_0(\lambda)\,\mathrm{e}^{2\pi i D_0}. \tag{41}$$

Solution about $\lambda = \infty$. Since $\lambda = \infty$ is an irregular singular point, the solution $\Phi(\lambda)$ of (36) in the neighbourhood of $\lambda = \infty$ has the formal expansion

$$\Phi_\infty = \hat{\Phi}_\infty\lambda^{D_\infty}\,\mathrm{e}^{Q(\lambda)} = (I + \Phi_{\infty 1}\lambda^{-1} + \cdots)\lambda^{D_\infty}\,\mathrm{e}^{Q(\lambda)}, \tag{42}$$

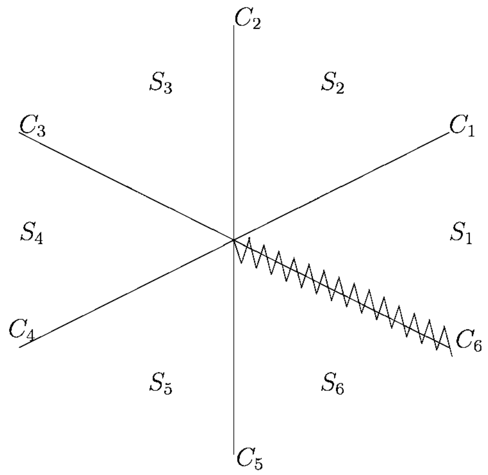

Figure 2.

where

$$D_\infty = \left(\alpha - \tfrac{1}{2}\right)\sigma_3, \qquad Q(\lambda) = -\left(\tfrac{2}{3}\lambda^3 + c_1\lambda^2 + x\lambda\right)\sigma_3. \tag{43}$$

The actual asymptotic behaviour of Φ changes in certain sectors of the complex λ-plane. These sectors are determined by $\mathrm{Re}\left(\frac{2}{3}\lambda^3 + c_1\lambda^2 + x\lambda\right) = 0$; thus for large λ the sectors are asymptotic to the rays $\arg\lambda = \frac{\pi}{6}(2j-3)$, $j = 1, 2, 3, 4, 5, 6$. Let $\Phi_j(\lambda)$, $j = 1, 2, 3, 4, 5, 6$, be solutions of (36) such that $\det\Phi_j(\lambda) = 1$ and $\Phi_j(\lambda) \sim \Phi_\infty$ as $|\lambda| \to \infty$ in the sector $S_j : \frac{\pi}{6}(2j-3) \leqslant \arg\lambda < \frac{\pi}{6}(2j-1)$ (see figure 2). Then the solutions $\Phi_j(\lambda)$ are related by the Stokes matrices, G_j, as follows:

$$\begin{aligned} &\Phi_{j+1}(\lambda) = \Phi_j(\lambda)G_j, \qquad j = 1, 2, 3, 4, 5, \\ &\Phi_1(\lambda) = \Phi_6(\lambda\,\mathrm{e}^{2\pi\mathrm{i}})G_6\,\mathrm{e}^{-2\pi\mathrm{i}D_\infty}, \end{aligned} \tag{44}$$

where

$$G_{2j-1} = \begin{pmatrix} 1 & a_{2j-1} \\ 0 & 1 \end{pmatrix}, \qquad G_{2j} = \begin{pmatrix} 1 & 0 \\ a_{2j} & 1 \end{pmatrix}, \qquad j = 1, 2, 3. \tag{45}$$

3.2. Monodromy Data

The relation between $\Phi_0(\lambda)$ and $\Phi_\infty(\lambda)$ is given by

$$\Phi_\infty(\lambda) = \Phi_0(\lambda)E_0, \tag{46}$$

where

$$E_0 = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \qquad \det(E_0) = 1. \tag{47}$$

The monodromy data $\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4\}$ satisfies the consistency condition

$$E_0^{-1}\,\mathrm{e}^{2\pi\mathrm{i}D_0}E_0\,\mathrm{e}^{2\pi\mathrm{i}D_\infty} = \prod_{j=1}^{6} G_j. \tag{48}$$

4. Schlesinger transformations

Let $\Phi(\lambda)$ be solution of (36) with parameters α, β and let $\Phi'(\lambda)$ be solution of (36) with parameters α', β'. We consider transformation

$$\Phi'(\lambda) = R(\lambda)\Phi(\lambda) \tag{49}$$

such that $\Phi'(\lambda)$ and $\Phi(\lambda)$ have the same monodromy data. Let $\alpha' = \alpha + n, \beta' = \beta + m$. Then (48) is invariant if $2n \pm m \in 2\mathbb{Z}$.

Let $R(\lambda) = R_j(\lambda)$ when $\lambda \in S_j$, $j = 1, 2, 3, 4, 5, 6$. Then (44) implies the following RH problem for $R(\lambda)$

$$\begin{aligned} &R_{j+1}(\lambda) = R_j(\lambda), \qquad &&\lambda \text{ on } C_{j+1}, \qquad j = 1, 2, 3, 4, 5,\\ &R_1(\lambda) = -R_6(\lambda\,\mathrm{e}^{2\pi \mathrm{i}}), \qquad &&\lambda \text{ on } C_1, \end{aligned} \tag{50}$$

with the boundary conditions

$$\begin{aligned} &R(\lambda) \sim \hat{\Phi}'_0 \lambda^{m\sigma_3} \hat{\Phi}_0^{-1}, \qquad &&\text{as } \lambda \to 0,\\ &R(\lambda) \sim \hat{\Phi}'_\infty \lambda^{n\sigma_3} \hat{\Phi}_\infty^{-1}, \qquad &&\text{as } |\lambda| \to \infty. \end{aligned} \tag{51}$$

All possible Schlesinger transformations admitted by equation (36) may be generated by the following transformations

$$\begin{cases} \alpha' = \alpha + \frac{1}{2} \\ \beta' = \beta + 1 \end{cases}, \qquad R_{(1)}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \lambda^{1/2} + \begin{pmatrix} \frac{1}{2} w r_1 & -\frac{1}{2} w \\ -r_1 & 1 \end{pmatrix} \lambda^{-1/2}, \tag{52}$$

$$\begin{cases} \alpha' = \alpha + \frac{1}{2} \\ \beta' = \beta - 1 \end{cases}, \qquad R_{(2)}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \lambda^{1/2} + \begin{pmatrix} \frac{1}{2} w r_2 & -\frac{1}{2} w \\ -r_2 & 1 \end{pmatrix} \lambda^{-1/2}, \tag{53}$$

$$\begin{cases} \alpha' = \alpha - \frac{1}{2} \\ \beta' = \beta + 1 \end{cases}, \qquad R_{(3)}(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \lambda^{1/2} + \begin{pmatrix} 1 & -\frac{1}{r_1} \\ -\frac{v}{2w} & \frac{v}{2wr_1} \end{pmatrix} \lambda^{-1/2}, \tag{54}$$

$$\begin{cases} \alpha' = \alpha - \frac{1}{2} \\ \beta' = \beta - 1 \end{cases}, \qquad R_{(4)}(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \lambda^{1/2} + \begin{pmatrix} 1 & -\frac{1}{r_2} \\ -\frac{v}{2w} & \frac{v}{2wr_2} \end{pmatrix} \lambda^{-1/2}, \tag{55}$$

where $r_1 = \frac{2H+\beta}{4wL}$ and $r_2 = \frac{2H-\beta}{4wL}$.

Let

$$\Phi'(\lambda, x; u', v', \alpha', \beta') = R_{(j)}(\lambda, x; u, v, \alpha, \beta)\Phi(\lambda, x; u, v, \alpha, \beta), \tag{56}$$

and

$$\Phi''(\lambda, x; u'', v'', \alpha'', \beta'') = R_{(i)}(\lambda, x; u', v', \alpha', \beta')\Phi'(\lambda, x; u', v', \alpha', \beta'). \tag{57}$$

Then

$$R_{(i)}(\lambda, x; u', v', \alpha', \beta')R_{(j)}(\lambda, x; u, v, \alpha, \beta) = I, \tag{58}$$

for $(i, j) = (3, 2)$ and $(i, j) = (1, 4)$. Moreover

$$R_{(1)}(\lambda, x; u', v', \alpha', \beta')R_{(2)}(\lambda, x; u, v, \alpha, \beta) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \lambda + \begin{pmatrix} -\frac{1}{2} u & -\frac{1}{2} w \\ \frac{2}{w} & 0 \end{pmatrix}. \tag{59}$$

The Schlesinger transformation (59) shifts the parameters as $\alpha' = \alpha + 1$, $\beta' = \beta$.

5. Bäcklund transformations

The linear equation (36*a*) is transformed under the Schlesinger transformations defined by the transformation matrices $R_{(j)}$, $j = 1, 2, 3, 4$ as follows:

$$\frac{\partial \Phi'}{\partial \lambda} = B'(\lambda)\Phi'(\lambda), \tag{60a}$$

$$B'(\lambda) = \left[R_{(j)}(\lambda)B(\lambda) + \frac{\partial}{\partial \lambda}R_{(j)}(\lambda)\right]R_{(j)}^{-1}(\lambda). \tag{60b}$$

Using (60*b*) we can derive the Bäcklund transformations between solutions $u(x)$ and $v(x)$ of (33), with parameters α and β, and solutions $u'(x)$ and $v'(x)$ of (33), with parameters α' and β'. The Bäcklund transformations corresponding to the Schlesinger transformations $R_{(j)}$, $j = 1, 2, 3, 4$ may be listed as follows:

$$\begin{aligned} R_{(1)}:\quad & v' = -\frac{1}{L^2}\left(H + \frac{1}{2}\beta\right)\left(uL + H + \frac{1}{2}\beta\right), \\ & u' = \frac{L(v - u_x)}{\left(uL + H + \frac{1}{2}\beta\right)} + \frac{1}{L}\left(uL + H + \frac{1}{2}\beta\right), \\ & \alpha' = \alpha + \frac{1}{2}, \qquad \beta' = \beta + 1, \end{aligned} \tag{61}$$

$$\begin{aligned} R_{(2)}:\quad & v' = -\frac{1}{L^2}\left(H - \frac{1}{2}\beta\right)\left(uL + H - \frac{1}{2}\beta\right), \\ & u' = \frac{L(v - u_x)}{\left(uL + H - \frac{1}{2}\beta\right)} + \frac{1}{L}\left(uL + H - \frac{1}{2}\beta\right), \\ & \alpha' = \alpha + \frac{1}{2}, \qquad \beta' = \beta - 1, \end{aligned} \tag{62}$$

$$\begin{aligned} R_{(3)}:\quad & v' = -\frac{L}{\left(H + \frac{1}{2}\beta\right)}\left[\frac{Lv^2}{\left(H + \frac{1}{2}\beta\right)} + v_x + uv\right], \\ & u' = -\frac{Lv}{\left(H + \frac{1}{2}\beta\right)} - \frac{\left(H + \frac{1}{2}\beta\right)}{L}, \\ & \alpha' = \alpha - \frac{1}{2}, \qquad \beta' = \beta + 1, \end{aligned} \tag{63}$$

$$\begin{aligned} R_{(4)}:\quad & v' = -\frac{L}{\left(H - \frac{1}{2}\beta\right)}\left[\frac{Lv^2}{\left(H - \frac{1}{2}\beta\right)} + v_x + uv\right], \\ & u' = -\frac{Lv}{\left(H - \frac{1}{2}\beta\right)} - \frac{\left(H - \frac{1}{2}\beta\right)}{L}, \\ & \alpha' = \alpha - \frac{1}{2}, \qquad \beta' = \beta - 1. \end{aligned} \tag{64}$$

5.1. Special solutions

In this section, we will derive special solutions for (33). The Bäcklund transformation (61) breaks down when $v = u_x$ and $uL + H + \frac{1}{2}\beta = 0$. Substituting $v = u_x$ into $uL + H + \frac{1}{2}\beta = 0$, we obtain

$$u_{xx} + (3u + 2c_1)u_x + u^3 + 2c_1u^2 + 2xu + \beta - 2\alpha + 1 = 0. \tag{65}$$

However u and v satisfy (33). This implies that α and β must satisfy $2\alpha+\beta+1=0$. Therefore we have shown that if $2\alpha+\beta+1=0$, then (33) admits special solution $v=u_x$ and u is a solution of the second Painlevé equation (65).

One can use the transformations (61)–(64) to obtain infinite hierarchies of rational solutions of (33). For example, if one start by the solution of (33)

$$v(x)=u(x)=0, \qquad \alpha=0, \qquad \beta=1, \tag{66}$$

then the transformation (61) yields the following solution of (33):

$$v'(x)=\frac{-1}{x^2}, \qquad u'(x)=\frac{1}{x}, \qquad \alpha'=\frac{1}{2}, \qquad \beta'=2. \tag{67}$$

Applying the transformation (61) to the solution (67), we obtain the solution

$$v''(x)=\frac{-2(x^2-c_1)}{x^2+c_1}, \qquad u''(x)=\frac{x^4-2c_1x^2-c_1^2}{x^3(x^2+c_1)}, \qquad \alpha''=1, \qquad \beta''=3. \tag{68}$$

If we apply the transformation (62) to the solution $\alpha=\alpha_0$, $\beta=-(2\alpha_0+1)$, $v=u_x$ and u is a solution of (65), then we obtain a new solution $\alpha'=\alpha_0+\frac{1}{2}$, $\beta'=-(2\alpha_0+2)$, $v'=u'_x$ and u' is a solution of the equation

$$u'_{xx}+(3u'+2c_1)u'_x+u'^3+2c_1u'^2+2xu'-4\alpha_0-2=0. \tag{69}$$

Applying the transformation (63) to the solution $\alpha=\alpha_0$, $\beta=-(2\alpha_0+1)$, $v=u_x$ and u is a solution of (65), then we obtain a new solution $\alpha'=\alpha_0-\frac{1}{2}$, $\beta'=-2\alpha_0$, $v'=u'_x$ and u' is a solution of the equation

$$u'_{xx}+(3u'+2c_1)u'_x+u'^3+2c_1u'^2+2xu'-4\alpha_0+2=0. \tag{70}$$

Thus we can obtain a hierarchy of special solutions $\alpha=\alpha_0+\frac{n}{2}$, $\beta=-(2\alpha_0+n+1)$, $n\in\mathbb{Z}$, $v=u_x$ and u is a solution of (65).

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